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Generalized basic hypergeometric functions and the q-analogues of 3-j and 6-j coefficients

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Abstract. The Racah-Wigner algebra for the *quantum group* $SU_q(2)$ is developed to derive explicit expressions for the q-analogues of the Van der Waerden, Racah, Wigner and Majumdar forms of the 3-j coefficient given in terms of sets of basic hypergeometric functions. Interrelationships between the members of a given set of ${}_3\phi_2$ are established using the *reversal* of series or the $q \to q^{-1}$ operation. Starting with the Van der Waerden set, using three transformations of ${}_3\phi_2$s, 12 other sets including the Racah, Wigner and Majumdar sets, have been obtained. In the simpler case of the q-analogue of the 6-j coefficients, two sets of ${}_4\phi_3$s, related to each other by *reversal* of series are obtained.

1. Introduction

Quantum deformations of Lie groups and Lie algebras or *quantum groups* (Sklyanin 1982, Kulish and Sklyanin 1982, Kulish and Reshetikhin 1982, Drinfeld 1986a, b, Jimbo 1986), which are strictly deformations of the universal enveloping algebra of an underlying Lie group, are of great importance for applications in diverse fields such as classical and quantum integrable systems, in quantum field theory, in statistical physics, and in the theory of basic hypergeometric functions. The quantum group $SU_q(2)$, which is a quantum deformation of SU(2), has been extensively studied (Sklyanin 1982, Kulish and Reshetikhin 1983, Jimbo 1986, Drinfeld 1986a, b, Kirillov and Reshetikhin 1988, Bo-Yu Hou *et al* 1989, MacFarlane 1989, Biedenharn 1989). The Racah-Wigner algebra for $SU_q(2)$ has been developed by Kirillov and Reshetikhin (1988) and by Bo-Yu Hou *et al* (1989).

The q-analogues of the Racah-Fock formulae for 3-j coefficients were first obtained by Vaksmann and Soibelman (1988). Other representations of the q-analogues of the 3-j coefficients—viz. the Van der Waerden and Majumdar formulae—as well as their symmetry properties were found by Kirillov and Reshetikhin (1988), and by Groza *et al* (1990)†. Kachurik and Klimyk (1990) have also given the q-analogue of the 6-j coefficient. These authors note that the q-analogues of the 3-j and the 6-j coefficients correspond to the basic hypergeometric functions ${}_3\phi_2(q)$ and ${}_4\phi_3(q)$, respectively. Bo-Yu Hou *et al* (1989) have computed in detail the explicit forms of the $q-3$-j and the $q-6$-j coefficients for the $SU_q(2)$ algebra, in agreement with the Kirillov-Reshetikhin forms but for changes in the definitions for the *basic* numbers. Bo-Yu Hou *et al* (1989) listed several explicit values for the $q-3$-j and the $q-6$-j coefficients,

† This will be referred to as reference I.

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besides proving the quantum Racah sum rule (see also Koelink and Koornwinder, 1989, Nomura, 1990).

Our aim in this article is to establish the full connection between the $q-3$-j and the $q-6$-j coefficients on the one hand and the basic generalized hypergeometric functions—viz. ${}_3\phi_2$ and ${}_4\phi_3$, respectively—on the other. Groza *et al* (1990) have recently studied the q-analogues of the well known classical expressions for Clebsch-Gordan coefficients of $U_q(SU_2)$ on the basis of the theory of basic hypergeometric functions. Kachurik and Klimyk (1990) obtained new expressions for the Racah coefficients of the quantum algebra $U_q(SU_2)$ also with the help of the results of the theory of basic hypergeometric functions. We present our results on $q-3$-j and the $q-6$-j coefficients from a different viewpoint. In the case of the 3-j and 6-j coefficients, Rajeswari and Srinivasa Rao (1989) and Srinivasa Rao *et al* (1975, 1977) showed that there exist four sets of ${}_3F_2(1)$s for the 3-j coefficient and two sets of ${}_4F_3(1)$s for the 6-j coefficient, respectively. Only one member of these sets were referred to commonly in the literature until then. In the present work, the q-generalizations of these sets of hypergeometric functions are obtained using the transformation theory of basic hypergeometric functions. In the case of the $q-3$-j coefficient, we obtain the q-analogues of the set of six ${}_3F_2(1)$s (Srinivasa Rao 1978) and show that there exist, for the Van der Waerden form, sets of ${}_3\phi_2$s corresponding to either the even or the odd permutations of the columns of

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q$$

and these subsets are related to one another by the *reversal* of series and/or the $q \to q^{-1}$ substitution. The complete schematic picture which emerges reveals interesting *structures*, viz. four sets of three ${}_3\phi_2$s which are related to one another either by *reversal* or $q \to q^{-1}$ substitution and these in the limit $q \to 1$ lead to the even/odd permutation counterparts of the set of six ${}_3F_2(1)$s. Starting from this highly symmetric Van der Waerden set of ${}_3\phi_2(q)$s for the $q-3$-j coefficient, 12 sets of ${}_3\phi_2(q)$s have been obtained with the help of three well known transformations for ${}_3\phi_2(q)$s. Each of the 12 sets contains 12 members. The results of reference I referred above correspond to seven members of this full realization of the connection between the $q-3$-j coefficients and ${}_3\phi_2(q)$s and these identifications are made as and when they arise.

The q-analogue of the 6-j coefficient is simpler (Kachurik and Klimyk 1990), mainly due to the expression being symmetric under the $q \to q^{-1}$ substitution. In this case the q-generalizations of our results for the 6-j coefficient lead to two sets of ${}_4\phi_3(q)$s—the set I contains three members and the set II contains four members—which are related to each other by the *reversal* of series. The use of the q-Bailey transformation on a member belonging to set I of the Saalschutzian ${}_4\phi_3(q)$ yields the result of Kachurik and Klimyk (1990). In the $q \to 1$ limit, we obtain the corresponding sets I and II of ${}_4F_3(1)$s derived by Srinivasa Rao *et al* (1975) and Srinivasa Rao and Venkatesh (1977), respectively. Particular cases of our results presented for the $q-3$-j and the $q-6$-j coefficients are contained in reference I and Kachurik and Klimyk (1990) respectively.

In section 2 we give the essential notation for the basic hypergeometric functions. In section 3, the required transformations and reversal formulae necessary in our study are given. In section 4, all the sets of ${}_3\phi_2$s for the $q-3$-j coefficient are derived and their interrelationships leading to the schematic picture are established. In section 5, the two sets of ${}_4\phi_3$s for the $q-6\text{-}j$ coefficient are derived and they are shown to be related to one another through the reversal of series. Section 6 summarizes the results and conclusions.

2. Notation

The quantum group $SU_q(2)$ of Sklyanin (1982, 1983, 1985), Jimbo (1985, 1986), Drinfeld (1986a, b) and Woronowicz (1987a, b) is a q-deformation of the Lie algebra SU(2) involving an indeterminate parameter q. The self-adjoint operators J_x, J_y, J_z satisfy the commutation relations

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = [2J_z] \tag{2}$$

where $J_\pm = J_x \pm iJ_y$ and the quantity within square brackets on the RHS of (2) is given by (4) or (5) below. However, the notation of Heine (1878) used in his study of basic hypergeometric functions is

$$[n]_q^{\mathrm{H}} = \frac{1-q^n}{1-q} \tag{3}$$

while Kirillov-Reshetikhin and the Soviet group use for the RHS of (2)

$$[n]_q^{\mathrm{R}} = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} \tag{4}$$

and Bo-Yu Hou *et al* use

$$[n]_q^{\mathrm{c}} = \frac{q^n-q^{-n}}{q-q^{-1}}. \tag{5}$$

Obviously, in (4) and (5) the $q \to q^{-1}$ symmetry is manifest. It is clear that the notations are interchangeable with the use of

$$[n]_q^{\mathrm{R}} = q^{-(n-1)/2}[n]_q^{\mathrm{H}} \tag{6}$$

and

$$[n]_q^{\mathrm{c}} \xrightarrow[q \to q^{1/2}]{} [n]_q^{\mathrm{R}} \tag{7}$$

where $n \in C$.

Throughout this article, we use only the Heine (1878) notation (3), since this is the one adopted in all the literature pertaining to basic hypergeometric functions (see Slater, 1966, Exton, 1983). Hence, we drop the indices on $[n]_q^{\mathrm{H}}$ and write simply $[n]$ to represent the RHS factor of (3).

The q-gamma function has the property

$$\Gamma_q(n) = [n-1]! = [n-1][n-2]\ldots[2][1] \qquad \text{with } [0]! = 1. \tag{8}$$

Jackson (1910) has shown that Γ_q satisfies the property

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{\omega}{S_q(\omega z)} \tag{9}$$

where

$$S_q(x+\omega) = -q^{-x/\omega} S_q(x). \tag{10}$$

From (9) and (10), it is straightforward to show that

$$\frac{\Gamma_q(z)\Gamma_q(1-z)}{\Gamma_q(z+n)\Gamma_q(1-z-n)} = (-1)^n q^{-nz-n(n-1)/2} \tag{11}$$

which is the q-analogue of

$$\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(z+n)\Gamma(1-z-n)} = (-1)^n. \tag{12}$$

3. Required transformations

The terminating generalized basic hypergeometric function (or series) is defined (Gasper and Rahman 1990) as

$$\begin{aligned} {}_{u+1}\phi_p &\left[\begin{matrix} \alpha_1, \alpha_2, \ldots \alpha_u, q^{-n} \\ \beta_1, \beta_2, \ldots \beta_p \end{matrix} ; q, z \right] \\ &= \sum_{r=0}^{n} \frac{[\alpha_1]_r [\alpha_2]_r \ldots [\alpha_u]_r [q^{-n}]_r}{[\beta_1]_r [\beta_2] \cdot\cdot [\beta_p]_r} [(-1)^r q^{r(r-1)/2}]^{p-u} \frac{z^r}{[q]_r} \end{aligned} \tag{13}$$

where $\alpha_1, \ldots, \alpha_u$ are the numerator parameters, the $(u+1)$th numerator parameter denoted by q^{-n} determines the terminating nature of the series and $\beta_1, \ldots, \beta_p$ are the denominator parameters. When $u = p$, the factor $[\ldots]^{p-u}$ becomes 1, and the definition (13) reduces to the one given in Bailey (1935) and Slater (1966). In (13) the notation of Watson is used in writing α in place of q^α, so that

$$\begin{aligned} [\alpha]_n &= (1-\alpha)(1-\alpha q)(1-\alpha q^2) \ldots (1-\alpha q^{n-1}) \\ &= \prod_{m=0}^{\infty} (1-\alpha q^m)/(1-\alpha q^{m+n}) \end{aligned} \tag{14}$$

for $n = 1, 2, \ldots$. Notice that

$$\lim_{q \to 1} \frac{[\alpha]_n}{[\beta]_n} = \frac{(\alpha)_n}{(\beta)_n} \tag{15}$$

where

$$(\alpha)_n = \left\{ \begin{matrix} \alpha(\alpha+1) \ldots (\alpha+n-1) & n = 1, 2, \ldots \\ 1 & n = 0 \end{matrix} \right\} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{16}$$

Conventionally, in the literature on basic hypergeometric functions, the Watson notation is adopted only for positive parameters, and a negative parameter is always written as q^{-n}. In this article, we depart from this convention and choose to use the Watson form for negative as well as positive parameters and, to make the termination obvious, when α is a negative parameter we write the q-analogue of the Pocchammer symbol as

$$[\alpha]_n = (1-q^\alpha)(1-q^{\alpha+1})(1-q^{\alpha+2}) \ldots (1-q^{\alpha+n-1}) \tag{17}$$

instead of (14). This notation enables us to write, for instance, the basic hypergeometric function part of the $q-3$-j coefficient as

$$ {}_3\phi_2(A, B, C; D, E; q, q) $$

which in the limit $q \to 1$ reduces to

$$ {}_3F_2(A, B, C; D, E; 1) $$

where A, B and C are negative parameters. This is aesthetically satisfying since, in our notation, the numerator and denominator parameters for the ${}_3\phi_2(q; q)$ and the ${}_3F_2(1)$ are one and the same.

It is straightforward to derive the reversal formula for a generalized basic hypergeometric series, when the first $s \leqslant p$ of the numerator parameters and the first $t \leqslant p$ denominator parameters are negative (with the termination being determined by the $(p+1)$th numerator parameter), as

$$ {}_{p+1}\phi_p\left[\begin{matrix} -\alpha_1, \ldots, -\alpha_s, \alpha_{s+1}, \ldots, \alpha_p, -n \\ -\beta_1, \ldots, -\beta_t, \beta_{t+1}, \ldots, \beta_p \end{matrix}; q, z\right] $$
$$ = (-1)^{n(s-t+1)} z^n q^{-n(n+1)/2} q^{-n(\alpha_1+\ldots+\alpha_s-\beta_1\ldots-\beta_t)+n(n-1)(s-t)} $$
$$ \times \frac{[1+\alpha_1-n]_n \ldots [1+\alpha_s-n]_n [\alpha_{s+1}]_n \ldots [\alpha_p]_n}{[1+\beta_1-n]_n \ldots [1+\beta_t-n]_n [\beta_{t+1}]_n \ldots [\beta_p]_n} $$
$$ \times {}_{p+1}\phi_p\left[\begin{matrix} 1+\beta_1-n, \ldots, 1+\beta_t-n, 1-\beta_{t+1}-n, \ldots, 1-\beta_p-n, -n \\ 1+\alpha_1-n, \ldots, 1+\alpha_s-n, 1-\alpha_{s+1}-n, \ldots, 1-\alpha_p-n \end{matrix}; q, \frac{1}{z} q^{\alpha_1+\ldots+\alpha_s-\alpha_{s+1}\ldots-\alpha_p-\beta_1\ldots-\beta_t+\beta_{t+1}\ldots+\beta_p+n+1}\right] \quad (18) $$

where the ${}_{p+1}\phi_p(q, z)$ is well defined only when the parameters satisfy the condition

$$ \min(\beta_1, \beta_2, \ldots, \beta_t) \geqslant \min(\alpha_1, \alpha_2, \ldots, \alpha_s) \geqslant n. \quad (19) $$

In the limit $q \to 1$, we get the generalization of the *reversal* formula for ${}_{p+1}F_p(z)$. It is a generalization in the sense that while the termination of the series is governed by the negative numerator parameter $-n$, it allows for s of the numerator (t of the denominator) parameters being negative. The generalized *reversal* formula (18) is precisely the one we require in our studies of $q-3$-j and $q-6$-j coefficients which have more than one of the numerator parameters being negative.

The q-generalization of the Weber and Erdelyi (1952) transformation I for ${}_3F_2(1)$ is given by Askey and Wilson (1985) as

$$ {}_3\phi_2\left[\begin{matrix} -n, \alpha, \beta \\ \gamma, \delta \end{matrix}; q, q\right] $$
$$ = q^{n\alpha} \Gamma_q[\gamma, \gamma+n-\alpha; \gamma+n, \gamma-\alpha]\, {}_3\phi_2\left[\begin{matrix} -n, \alpha, \delta-\beta \\ \delta, 1+\alpha-\gamma-n \end{matrix}; q, q^{1+\beta-\gamma}\right] \quad (20) $$

where we have used the notation

$$ \Gamma_q[x, y, \ldots; a, b, \ldots] = \frac{\Gamma_q[x]\Gamma_q[y]\ldots}{\Gamma_q[a]\Gamma_q[b]\ldots}. \quad (21) $$

In the limit $q \to 1$, we have

$$ {}_3F_2\left[\begin{matrix}-n, \alpha, \beta\\ \gamma, \delta\end{matrix}; 1\right] = \Gamma(\gamma, \gamma+n-\alpha; \gamma+n, \gamma-\alpha)\,{}_3F_2\left[\begin{matrix}-n, \alpha, \delta-\beta\\ \delta, 1+\alpha-\gamma-n\end{matrix}; 1\right] \tag{22} $$

where we have used the notation

$$ \Gamma(x, y, \ldots; a, b, \ldots) = \frac{\Gamma(x)\Gamma(y)\ldots}{\Gamma(a)\Gamma(b)\ldots}. \tag{23} $$

Use of the above Weber-Erdelyi transformation I for ${}_3F_2(1)$ recursively, with γ and δ interchanged, results in the Weber-Erdelyi transformation II given by

$$ {}_3F_2\left[\begin{matrix}-n, \alpha, \beta\\ \gamma, \delta\end{matrix}; 1\right] = \frac{(\gamma-\alpha)_n(\delta-\alpha)_n}{(\gamma)_n(\delta)_n}\,{}_3F_2\left[\begin{matrix}-n, \alpha, 1-s\\ 1+\alpha-\gamma-n, 1+\alpha-\delta-n\end{matrix}; 1\right] \tag{24} $$

with the usual notation for the Pocchammer symbol (16) and

$$ s = \gamma+\delta-\alpha-\beta+n. \tag{25} $$

A recursive use cannot be made of (20) as such, since the LHS ${}_3\phi_2(q)$ is a polynomial in q while the RHS ${}_3\phi_2(q^{1+\beta-\gamma})$ is a polynomial in $q^{1+\beta-\gamma}$. If (20) were a transformation given in terms of the general variable z, then it could have been used recursively. What we have in (20) is a transformation for the particular case $z=q$. Thus, (20) is *a* q-generalization of (22) and not *the* q-generalization of (22). However, after the reversal of the ${}_3\phi_2(q^{1+\beta-\gamma})$ on the RHS of (20), we get

$$ \begin{aligned} {}_3\phi_2\left[\begin{matrix}-n, \alpha, \beta\\ \gamma, \delta\end{matrix}; q, q\right] &= \Gamma_q[\alpha+n, \delta-\beta+n, \gamma, \delta; \alpha, \delta-\beta, \gamma+n, \delta+n] \\ &\quad \times {}_3\phi_2\left[\begin{matrix}-n, \gamma-\alpha, 1-\delta-n\\ 1-\alpha-n, 1+\beta-\delta-n\end{matrix}; q, q\right] \end{aligned} \tag{20$'$} $$

and now (20′) can be iterated. One such iteration results in a ${}_3\phi_2(q^{1-\beta})$, which on reversal can be shown to yield (27).

The transformation (20) can be obtained as a special case of a transformation given by Sears (1951) for balanced ${}_4\phi_3(q)$s, viz.

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix}-n, a, b, c\\ d, e, f\end{matrix}; q, q\right] &= q^{na}\frac{[e-a]_n[f-a]_n}{[e]_n[f]_n}\,{}_4\phi_3\left[\begin{matrix}-n, a, d-b, d-c\\ d, 1+a-e-n, 1+a-f-n\end{matrix}; q, q\right] \end{aligned} \tag{26} $$

whose parameters obey the Saalschutz condition

$$ 1-n+a+b+c = d+e+f $$

which in Watson's notation is: $q^{1-n}abc = def$.

Letting $c, f \to 0$, yields after some simplification (20). If we let $c, d \to 0$, then we get

$$ {}_3\phi_2\left[\begin{matrix}-n, a, b\\ e, f\end{matrix}; q, q\right] = \frac{[e-a]_n[f-a]_n}{[e]_n[f]_n}\,q^{na}\,{}_3\phi_2\left[\begin{matrix}-n, a, 1+a+b-e-f-n\\ 1+a-e-n, 1+a-f-n\end{matrix}; q, q\right] \tag{27} $$

which is a q-analogue of the Weber-Erdelyi transformation (II) given by (24).

If we let $a, f \to 0$, then we get

$$ {}_3\phi_2\left[\begin{matrix}-n, b, c\\ d, e\end{matrix}; q, q\right] = \frac{[d+e-b-c]_n}{[e]_n} q^{n(b+c-d)} {}_3\phi_2\left[\begin{matrix}-n, d-b, d-c\\ d, d+e-b-c\end{matrix}; q, q\right] \tag{28} $$

which is a terminating q-analogue of the Kummer-Thomae-Whipple-formula (see Gasper and Rahman, 1990: (3.2.8), p 61), which in the limit $q \to 1$ yields the transformation

$$ {}_3F_2\left[\begin{matrix}-n, b, c\\ d, e\end{matrix}; 1\right] = \frac{(d+e-b-c)_n}{(e)_n} {}_3F_2\left(\begin{matrix}-n, d-b, d-c\\ d, d+e-b-c\end{matrix}; 1\right). \tag{29} $$

The *reversal* and the other transformation formulae given above are used in our study of the sets of generalized basic hypergeometric functions and the q-analogues of 3-j and 6-j coefficients.

4. q-analogues of the 3-j coefficients

The starting point for us is the q-analogue of the Van der Waerden form of the 3-j coefficients given explicitly by Kirillov and Reshetikhin (1988) and others, as

$$ \begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{\mathrm{KR}} &= (-1)^{j_1-j_2-m_3}[2j_3+1]^{-1/2} q^{(m_2-m_1)/6} C^{j_1 j_2}_{m_1 m_2, j_3 - m_3} \\ &= (-1)^{j_1-j_2-m_3} q^{(1/4)\beta_3(J+1)+(1/2)(j_1 m_2 - j_2 m_1)+(1/6)(m_2-m_1)} \Delta_{\mathrm{R}}(j_1 j_2 j_3) \\ &\quad \times \left(\prod_{i=1}^{3} [j_i+m_i]![j_i-m_i]!\right)^{1/2} \\ &\quad \times \sum_n (-1)^n q^{-(n/2)(J+1)} \left([n]! \prod_{k=1}^{2} [n-\alpha_k]! \prod_{l=1}^{3} [\beta_l - n]!\right)^{-1} \end{aligned} \tag{30} $$

where

$$ \Delta_{\mathrm{R}}(j_1 j_2 j_3) = \left(\frac{[-j_1+j_2+j_3]![j_1-j_2+j_3]![j_1+j_2-j_3]!}{[j_1+j_2+j_3+1]!}\right)^{1/2} \tag{31} $$

$$ \alpha_1 = j_1 - j_3 + m_2 \qquad \alpha_2 = j_2 - j_3 - m_1 \qquad \beta_1 = j_1 - m_1 $$

$$ \beta_2 = j_2 + m_2 \qquad \beta_3 = j_1 + j_2 - j_3 \tag{32} $$

$$ \max(0, \alpha_1, \alpha_2) \leqslant n \leqslant \min(\beta_1, \beta_2, \beta_3) \tag{33} $$

with

$$ m_1 + m_2 + m_3 = 0 \qquad \text{and} \qquad J = j_1 + j_2 + j_3. \tag{34} $$

In the expressions (30) and (31) above, all the factors are in the Kirillov-Reshetikhin notation, viz. (4). To enable us to write (30) in terms of a ${}_3\phi_2(q)$, we change over into the Heine notation (3), and after simplification using the relevant definitions given in section 2, we get finally

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q$$

$$= (-1)^{\alpha_1-\alpha_2} q^{(1/2)(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1)+(1/3)(\beta_1+\beta_2+\beta_3)-(1/6)(\alpha_1+\alpha_2)}$$
$$\times\left(\prod_{i=1}^{2}\prod_{j=1}^{3}[\beta_j-\alpha_i]![\beta_j]!/[\beta_1+\beta_2+\beta_3-\alpha_1-\alpha_2+1]!\right)^{1/2}$$
$$\times(\Gamma_q(1-\alpha_1, 1-\alpha_2, 1+\beta_1, 1+\beta_2, 1+\beta_3))^{-1}$$
$$\times {}_3\phi_2(-\beta_1, -\beta_2, -\beta_3; 1-\alpha_1, 1-\alpha_2; q; q). \tag{35}$$

This van der Waerden form of the $q-3$-j coefficient is manifestly invariant under the 3! permutations of $\beta_1, \beta_2, \beta_3$ and the 2! permutations of α_1, α_2. Thus, it exhibits 12 symmetries of the $q-3$-j coefficient. These are, however, not the 12 symmetries which arise due to the column permutations of the $q-3$-j symbol and $m_i \to -m_i$. To account for the 72 symmetries exhibited by this coefficient when it is represented as the $q-3\times 3$ square symbol of Regge (1958),

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{Vmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_3 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{Vmatrix} = \|R_{ik}\| \tag{36}$$

it is necessary to obtain five other ${}_3\phi_2$s for the $q-3$-j coefficient. The presence of the q-factor inside the summation on the RHS of (30) clearly reveals (i) the non-invariance of (30) under $q \to q^{-1}$ substitution and (ii) it contributes to the separation of the set of six ${}_3\phi_2$s into two sets of three ${}_3\phi_2$s which correspond to the even and odd permutations of the columns of

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q.$$

As in the case of the 3-j coefficient (Srinivasa Rao 1978) the required five series representations (or ${}_3\phi_2$s) are obtained by replacing the summation index n in (30) by $n-\alpha_k (k=1, 2)$ and $\beta_l - n (l=1, 2, 3)$. Of these, the $n-\alpha_k (k=1, 2)$ substitutions along with (30) will give rise to the set of three ${}_3\phi_2(q)$s:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q$$

$$= (-1)^{\sigma(rst)} q^P \left(\prod_{i,k=1}^{3}[R_{ik}]!/[J+1]!\right)^{1/2}$$
$$\times(\Gamma_q(1-A, 1-B, 1-C, D, E))^{-1}\, {}_3\phi_2\left[\begin{matrix} A, B, C \\ D, E \end{matrix}; q, q\right] \tag{37}$$

where

$$A=-R_{2r} \qquad B=-R_{3s} \qquad C=-R_{1t}$$
$$D=1+R_{3t}-R_{2r} \qquad E=1+R_{2t}-R_{3s} \tag{38}$$
$$\sigma(rst)=R_{3r}-R_{2s} \qquad P=\tfrac{1}{2}(AB+BC+CA)-\tfrac{1}{3}(A+B+C)+\tfrac{1}{6}(D+E-2)$$

for even permutations of $(rst)=(123)$. For the permutation $(rst)=(123)$ in (37) and (38), we obtain (43) of reference I.

The substitutions $\beta_l - n(l=1,2,3)$ in (30) give rise to

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = (-1)^{\sigma(rst)} q^P \left(\prod_{i,k=1}^{3} [R_{ik}]!/[J+1]! \right)^{1/2} \times (\Gamma_q(1-A', 1-B', 1-C', D', E'))^{-1} {}_3\phi_2\left[\begin{matrix} A', B', C' \\ D', E' \end{matrix}; q, q^{s'}\right] \tag{39}$$

where

$$A' = -R_{2s} \qquad B' = -R_{3r} \qquad C' = -R_{1t} \qquad D' = 1 + R_{3t} - R_{2s}$$
$$E' = 1 + R_{2t} - R_{3r} \qquad \sigma(rst) = R_{3r} - R_{2s} + J \tag{40}$$
$$P = \tfrac{1}{2}(1-D')(1-E') - \tfrac{1}{6}(A'+B'+C') + \tfrac{1}{3}(D'+E'-2).$$

and $s' = J+2$, for even permutations of $(rst)=(123)$. Interchanging $r \leftrightarrow s$ in A', B', C', D', E' of (40) gives A, B, C, D, E given by (38) for odd permutations of (123). It is customary to call the basic hypergeometric series in q as type I and the series in $q^{s'}(s' = J+2 = D'+E'-A'-B'-C')$ as type II. Since we now have ${}_3\phi_2$s occurring in (37) and (39) being considered as sets of three corresponding to even and odd permutations of $(rst)=(123)$, respectively we introduce the notation ${}_3^{\mathrm{I}}\phi_2^{\mathrm{e}}(q)$ and ${}_3^{\mathrm{II}}\phi_2^{\mathrm{o}}(q^s)$ to denote these sets.

We notice that the *reversal* formula (18) precisely takes a polynomial in z to a polynomial in q^{1+s}/z and, for the special case $z=q$, (18) gives a relationship between a ${}_{p+1}\phi_p(q,q)$ and a ${}_{p+1}\phi_p(q,q^s)$. Therefore, starting with (37) and (38) if we use the reversal formula (18), we will arrive at (39) and (40). The question arises as to whether these two sets provide the q-generalization for the set of six ${}_3F_2(1)$s. For, they belong to two different types being polynomials in q and q^s, respectively. Also, Bo-Yu Hou *et al* (1989) have shown that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \neq \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}} \tag{41}$$

but, instead

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_q. \tag{42}$$

This is due to lack of symmetry of (30) under $q \to q^{-1}$. In the set of three ${}_3^{\mathrm{I}}\phi_2^{\mathrm{e}}(q)$ given by (37), (38), if we substitute $q \to q^{-1}$, we get

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}} = (-1)^{\sigma(rst)} q^P \left(\prod_{i,k=1}^{3} [R_{ik}]!/[J+1]! \right)^{1/2} \times (\Gamma_q(1-A, 1-B, 1-C, D, E))^{-1} {}_3\phi_2\left[\begin{matrix} A, B, C \\ D, E \end{matrix}; q, q^{s}\right] \tag{43}$$

where $\sigma(rst)$ and $s=J+2$ are as in (38) but

$$P=-\tfrac{1}{6}(A+B+C)-\tfrac{1}{6}(D+E+1)+\frac{DE}{2}. \tag{44}$$

This set of three series corresponding to even permutations of $(rst)=(123)$ will be denoted by ${}^{\rm II}_{3}\phi^{e}_{2}(q^{s})$. These along with the set of ${}^{\rm II}_{3}\phi^{o}_{2}(q^{s})$ given in (39), (40) constitute the set of six ${}_3\phi_2(q^{s})$s which provide a q-generalization of the set of six ${}_3F_2(1)$ functions.

In the set of three ${}^{\rm II}_{3}\phi^{o}_{2}(q^{s})$ functions given by (39), (40), if we substitute $q\to q^{-1}$, we get

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}}$$

$$=(-1)^{\sigma(rst)}q^{P}\left(\prod_{i,k=1}^{3}[R_{ik}]!/[J+1]!\right)^{1/2}$$

$$\times(\Gamma_q(1-A,1-B,1-C,D,E))^{-1}\,{}_3\phi_2\left[\begin{matrix}A,B,C\\D,E\end{matrix};q,q\right] \tag{37$'$}$$

where $\sigma(rst)$ is as in (40) and P is as in (38). Notice that (37′) differs from (37) only through a phase factor, consistent with (42). This set of three series representations correspond to odd permutations of $(rst)=(123)$ and will be denoted by ${}^{\rm I}_{3}\phi^{o}_{2}(q)$, which along with the set of three ${}^{\rm I}_{3}\phi^{e}_{2}(q)$ given by (37), (38) constitute the set of six ${}_3\phi_2(q)$ functions.

The two sets of three functions ${}^{\rm I}_{3}\phi^{o}_{2}(q)$ and ${}^{\rm II}_{3}\phi^{e}_{2}(q^{s})$ can be shown to be related to one another by *reversal* of series. The interconnection between the four sets of three ${}_3\phi_2$s is given in the schematic figure 1.

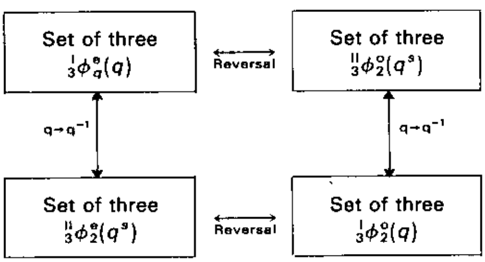

Figure 1. Interconnections between the four sets of three ${}_3\phi_2$ functions (defined in the text).

In the limit $q\to 1$, the above-described sets of ${}_3\phi_2$ functions will reduce to two sets of three ${}_3F_2(1)$ functions which correspond to even or odd permutations of the columns of the 3-j coefficient. This is schematically shown in figure 2.

Thus, either the three ${}^{\rm I}_{3}\phi^{e}_{2}(q)$ functions and the three ${}^{\rm I}_{3}\phi^{o}_{2}(q)$ functions, or the three ${}^{\rm II}_{3}\phi^{e}_{2}(q^{s})$ functions and the three ${}^{\rm II}_{3}\phi^{o}_{2}(q^{s})$ functions, constitute equivalent q-generalizations of the set of six ${}_3F_2(1)$ functions for the 3-j coefficient and their interrelationships are as in figure 1.

Starting with the Van der Waerden set of six ${}_3\phi_2$ functions (belonging to set I or set II corresponding to series expansions in q or q^{s}, respectively), using the transformations (20), (27) and (28), we get different formulae for the $q-3$-j coefficient. First, we

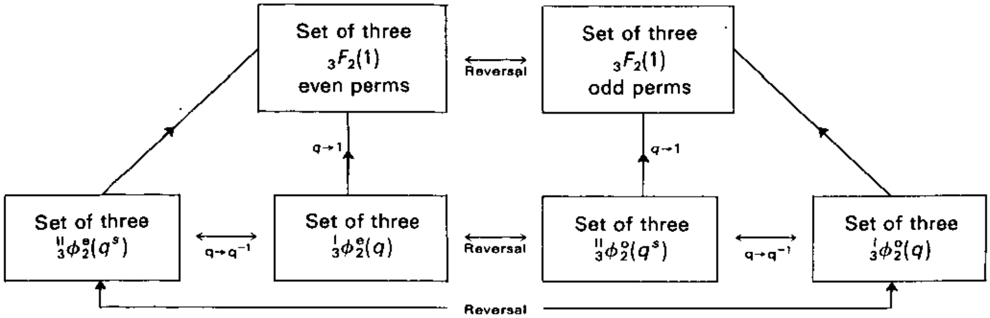

Figure 2. Set of six $_3F_2(1)$s and their q-generalizations.

use (20) to obtain the q-generalizations of the Wigner, Racah and Majumdar forms for the $q-3$-j coefficient. These forms are derived by simply using the q-analogue of the Erdelyi–Weber transformation, given by (20), on (37) and (37′), in three different ways. Corresponding to (37), the general form for the $q-3$-j coefficient thus obtained is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q$$

$$= \delta_{m_1+m_2+m_3,0}(-1)^{\sigma(rst)} q^P \left(\prod_{i,k=1}^{3} [R_{ik}]!/[J+1]! \right)^{1/2}$$

$$\times \Gamma_q[1-D'; 1-A', 1-C', E', 1+A'-D', 1+B'-E', 1+C'-D']$$

$$\times {}_3\phi_2 \left[\begin{matrix} A', B', C' \\ D', E' \end{matrix} ; q, q^{\varepsilon} \right] \tag{45}$$

with A', B', C', D',E' as in table 1, $\varepsilon = s' = D'+E'-A'-B'-C'$ and

$$P = \tfrac{1}{2}[(E'-B')(A'+C') - A'C'] + \tfrac{1}{6}(2B'-A'-C'-D'-E'-1). \tag{46}$$

This set corresponds to $^{\mathrm{II}}_3\phi_2^{\mathrm{e}}(q^s)$ in our notation, since (37) and hence (45) is for even permutations of $(rst) = (123)$. Similarly, corresponding to (37′) we would get a general expression (45′) which differs from (45) only in that the LHS of (45′) would be

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}}$$

and $\sigma(rst)$ would be as in (40) for odd permutations of $(rst) = (123)$ and P is as in (46). (We have not written down (45′), explicitly.)

This set given by (45′), (46′) corresponds to $^{\mathrm{II}}_3\phi_2^{\mathrm{o}}(q^s)$ in our notation. Thus, the Weber–Erdelyi transformation (20) when applied to (37), (37′) results in the transformation of the set of Van der Waerden $^{\mathrm{I}}_3\phi_2^{\mathrm{e,o}}(q)$ into the Racah, Wigner or Majumdar set of $^{\mathrm{II}}_3\phi_2^{\mathrm{e,o}}(q^s)$.

Here, it is to be noted that (20) is a q-analogue of the Weber–Erdelyi transformation 1 and not the *exact* analogue, since (20) is not for

$${}_3\phi_2 \left[\begin{matrix} -n, \alpha, \beta \\ \gamma, \delta \end{matrix} ; q, z \right]$$

Table 1. Use of the q-analogue of the Weber–Erdelyi transformation I given by (20) results in the expression (45). Column 1 refers to the use of (20), column 2 gives the numerator and denominator parameters of the ${}_3\phi_2$ in (45) and column 3, the identification of the result of the use of (20) on (37) and (37′).

Parameters in (20)	Parameters in (45)	Identification of (45)
$-n=-R_{3s}$	$A'=-R_{2r}$	q-Racah
$\alpha=-R_{2r}$	$B'=1+R_{3r}$	${}^{II}_{3}\phi_2^{e,o}(q^{s'})$
$\beta=-R_{1t}$	$C'=-R_{3s}$	
$\gamma=1+R_{3t}-R_{2r}$	$D'=-R_{3s}-R_{3t}$	
$\delta=1+R_{2t}-R_{3s}$	$E'=1+R_{2t}-R_{3s}$	
$-n=-R_{3s}$	$A'=-R_{3s}$	q-Majumdar
$\alpha=-R_{1t}$	$B'=1+R_{1s}$	${}^{II}_{3}\phi_2^{e,o}(q^{s'})$
$\beta=-R_{2r}$	$C'=-R_{1t}$	
$\gamma=1+R_{3t}-R_{2r}$	$D'=-R_{2s}-R_{3s}$	
$\delta=1+R_{2t}-R_{3s}$	$E'=1+R_{2t}-R_{3s}$	
$-n=-R_{1t}$	$A'=-R_{1t}$	q-Wigner
$\alpha=-R_{2r}$	$B'=1+R_{2t}$	${}^{II}_{3}\phi_2^{e,o}(q^{s'})$
$\beta=-R_{3s}$	$C'=-R_{2r}$	
$\gamma=1+R_{3t}-R_{2r}$	$D'=-R_{1t}-R_{3t}$	
$\delta=1+R_{2t}-R_{3s}$	$E'=1+R_{2t}-R_{3s}$	

From the ${}^{II}_{3}\phi_2^{e,o}(q^{s'})$ forms by *reversal* ${}^{I}_{3}\phi_2^{o,e}(q)$ and by $q\to q^{-1}\,{}^{I}_{3}\phi_2^{e,o}(q)$ are obtained, as given by the schematic diagram, figure 1. *Note*: As in the case of Rajeswari and Srinivasa Rao (1989), the identification of γ as $1+R_{2t}-R_{3s}$ and δ as $1+R_{3t}-R_{2r}$ results only in the same three sets but in a different order, namely q-Racah, q-Wigner and q-Majumdar forms.

but only for

$$ {}_3\phi_2\left[\begin{matrix}-n,\alpha,\beta\\ \gamma,\delta\end{matrix};q,q\right]. $$

Due to this reason (as stated in section 3), we cannot apply (20) directly to the Van der Waerden sets of three ${}^{II}_{3}\phi_2^{e,o}(q^{s})$ given by (39) and (43). However, use of *reversal* formula on the Racah, Wigner or Majumdar set of ${}^{II}_{3}\phi_2^{e}(q^{s})$ and ${}^{II}_{3}\phi_2^{o}(q^{s})$ given by (45) and (45′) results in the corresponding ${}^{I}_{3}\phi_2^{o}(q)$ and ${}^{I}_{3}\phi_2^{e}(q)$, respectively—the resultant expressions after algebraic simplifications can be shown to be the same in form as (45) and (45′) but with $\varepsilon=1$ and

$$ P=\tfrac{1}{2}E'(A'+C'-D')+\tfrac{1}{6}(B'+D'+E'-2A'-2C'-2). \tag{46′} $$

Equivalently, use of $q\to q^{-1}$ on the Racah, Wigner or Majumdar set of ${}^{II}_{3}\phi_2^{e,o}(q^{s})$ results in ${}^{I}_{3}\phi_2^{e,o}(q)$, as per the schematic diagram of figure 1. Thus, the Racah, Wigner, and Majumdar set of six ${}^{I}_{3}\phi_2^{e,o}(q)$ and the equivalent set of six ${}^{II}_{3}\phi_2^{e,o}(q^{s})$ can be generated from the corresponding Van der Waerden sets with the use of the Erdelyi–Weber transformation I.

Table 1 summarizes the identifications (column 1) to be made in (20) and the resulting numerator and denominator parameters (column 2) for the ${}_3\phi_2$s in expression (45). In column 3 are given identifications of the expression (45) as the q-analogues of the Racah, Wigner or Majumdar sets of ${}_3\phi_2$ functions. We can identify (41*b*), (42) and (46) of reference I to correspond to $(rst)=(132)$ in the q-Racah ${}^{I}_{3}\phi_2^{o}(q)$ set, to

$(rst) = (123)$ in the q-Racah ${}^{\text{II}}_{3}\phi^{\text{e}}_{2}(q^{s})$ set and to $(rst) = (321)$ in the q-Wigner ${}^{\text{I}}_{3}\phi^{\text{o}}_{2}(q)$ set, respectively.

The use of the transformation (27)—viz. the q-analogue of the Weber–Erdelyi transformation II, which exhibits manifestly $e \leftrightarrow f$ symmetry—on (37) results in the general expression

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \\
&\quad = \delta_{m_1+m_2+m_3,0}(-1)^{\sigma(rst)} q^{P} \left(\prod_{i,k=1}^{3} [R_{ik}]!/[J+1]! \right)^{1/2} \\
&\quad \times \Gamma_q[1-D', 1-E'; 1-A', 1-B', 1+A'-D', \\
&\qquad 1+A'-E', 1+B'-D', 1+B'-E', s'] \\
&\quad \times {}_3\phi_2\left[\begin{matrix} A', B', C' \\ D', E' \end{matrix}; q, q^{\varepsilon}\right]
\end{aligned} \tag{47}$$

with A', B', C', D', E' as in table 2, $\varepsilon = 1$, and

$$P = \tfrac{1}{2}[-A'B' + (1-s')(A'+B')] - \tfrac{1}{3}(A'+B'+C') + \tfrac{1}{6}(D'+E'-2) \tag{48}$$

for even permutations of $(rst) = (123)$; which set we denote by ${}^{\text{I}}_{3}\phi^{\text{e}}_{2}(q)$. Similarly, corresponding to (37′) we would get a general expression (47′), which differs from (47) only in that its LHS would be

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}}$$

Table 2. Use of the q-analogue of the Weber–Erdelyi transformation II given by (27) results in the expression (47). Column 1 refers to the use of (27), column 2 gives the numerator and denominator parameters of the ${}_3\phi_2$ in (47) and column 3 identifies the result of the use of (27) on (37) and (37′).

Parameters in (27)	Parameters in (47)	Identification of (47)
$-n = -R_{2r}$	$A' = -R_{2r}$	${}^{\text{I}}_{3}\phi^{\text{e,o}}_{2}(q)$
$a = -R_{3s}$	$B' = -R_{3s}$	$(rst) = (123)$ corresponds to the
$b = -R_{1t}$	$C' = -J - 1$	q-analogue of (26) of Raynal
$e = 1 + R_{3t} - R_{2r}$	$D' = -R_{3s} - R_{3t}$	
$f = 1 + R_{2t} - R_{3s}$	$E = -R_{2r} - R_{2t}$	
$-n = -R_{3s}$	$A' = -R_{3s}$	${}^{\text{I}}_{3}\phi^{\text{e,o}}_{2}(q)$
$a = -R_{1t}$	$B' = -R_{1t}$	$(rst) = (123)$ corresponds to the
$b = -R_{2r}$	$C' = -J - 1$	q-analogue of (27) of Raynal
$e = 1 + R_{3t} - R_{2r}$	$D' = -R_{2s} - R_{3s}$	
$f = 1 + R_{2t} - R_{3s}$	$E' = -R_{3t} - R_{3s}$	
$-n = -R_{1t}$	$A' = -R_{2r}$	${}^{\text{I}}_{3}\phi^{\text{e,o}}_{2}(q)$
$a = -R_{2r}$	$B' = -R_{1t}$	$(rst) = (123)$ and $m_i \to -m_i$
$b = -R_{3s}$	$C' = -J - 1$	corresponds to the q-analogue of
$e = 1 + R_{3t} - R_{2r}$	$D' = -R_{1t} - R_{3t}$	(27) of Raynal
$f = 1 + R_{2t} - R_{3s}$	$E' = -R_{1s} - R_{1t}$	

and $\sigma(rst)$ would be as in (40) for odd permutations of $(rst)=(123)$—(we do not write down (47′) explicitly)—which we denote by ${}^{\mathrm{I}}_{3}\phi^{\mathrm{o}}_{2}(q)$. Use of $q\to q^{-1}$ on these ${}^{\mathrm{I}}_{3}\phi^{\mathrm{e,o}}_{2}(q)$ sets will result in ${}^{\mathrm{II}}_{3}\phi^{\mathrm{e,o}}_{2}(q^{s'})$ and the expressions can be shown (after simplification) to be the same as (47), (47′), except for $\varepsilon=s'$ and

$$P=\frac{A'}{2}(A'+1)+\frac{B'}{2}(B'+1)+\tfrac{1}{2}[D'E'-(A'+B')(D'+E')]$$
$$-\tfrac{1}{6}(A'+B'+C'+D'+E'+1). \tag{48′}$$

Table 2 summarizes the identifications to be made in (27) and the resulting numerator and denominator parameters for the ${}_3\phi_2$ in expression (47) or (47′). In column 3 of this table are given the identification of a member of the set (47) or (447′), as the q-analogue of the form corresponding to $(rst)=(123)$, given by Raynal (1978). The first and third entries in this table can be identified with equations (44) and (45) of reference I. They are obtained by setting for the parameters (given in column 2 of table 2) $(rst)=(132)$ in ${}^{\mathrm{II}}_{3}\phi^{\mathrm{o}}_{2}(q^{s'})$ and $(rst)=(321)$ in ${}^{\mathrm{II}}_{3}\phi^{\mathrm{o}}_{2}(q^{s'})$, respectively.

The use of transformation (28)—viz. the terminating q-analogue of the Kummer–Thomae–Whipple formula, which exhibits manifestly $b\leftrightarrow c$ symmetry—on (37) results in

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}_q$$
$$=\delta_{m_1+m_2+m_3,0}(-1)^{\sigma(rst)}q^P\left(\prod_{i,k=1}^{3}[R_{ik}]!/[J+1]!\right)^{1/2}$$
$$\times\Gamma_q[E'-A';1-A',1+B'-D',1+C'-D',D',E',s']$$
$$\times{}_3\phi_2\begin{bmatrix}A',B',C'\\ D',E'\end{bmatrix};q,q^{\varepsilon}\Bigg] \tag{49}$$

where A', B', C', D', E' are as in table 3, $\varepsilon=1$ and

$$P=\tfrac{1}{2}(D'-B')(D'-C')+\frac{A'}{2}(B'+C')-\tfrac{1}{3}(A'+D'+1)+\tfrac{1}{6}(B'+C'-E') \tag{50}$$

for even permutations of $(rst)=(123)$; which set we denote by ${}^{\mathrm{I}}_{3}\phi^{\mathrm{e}}_{2}(q)$. Similarly, corresponding to (37′) we would get a general expression (49′), which differs from (49) only in that its LHS would be

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}_{q^{-1}}$$

and $\sigma(rst)$ would be as in (40) for odd permutations of $(rst)=(123)$ (we do not write down (49′) explicitly) which we denote by ${}^{\mathrm{I}}_{3}\phi^{\mathrm{o}}_{2}(q)$. Use of $q\to q^{-1}$ on these ${}^{\mathrm{I}}_{3}\phi^{\mathrm{e,o}}_{2}(q)$ sets will result in ${}^{\mathrm{II}}_{3}\phi^{\mathrm{e,o}}_{2}(q^{s'})$ and the expressions can be shown (after simplification) to be the same as (49), (49′), except for $\varepsilon=s'$ and

$$P=\tfrac{1}{2}[(1-D')(1-s')+A'D']-\tfrac{1}{6}(4A'+B'+C')+\tfrac{1}{3}(D'+E'-2). \tag{50′}$$

Table 3 summarizes the identifications to be made in (28) and the resulting numerator and denominator parameters for the ${}_3\phi_2$ in expression (49) or (49′). In column 3 are given the identification of a member of the set (49) or (49′), as the q-analogue of the form corresponding to $(rst)=(123)$ given by Raynal (1978). Equation

Table 3. Use of the q-analogue of the Kummer–Thomae–Whipple transformation (28) results in the expression (49). The parameters in (28) and (49) are given in columns 1 and 2, and column 3 gives the identification of the result of the use of (28) on (37) and (44). The expression (49) corresponds to ${}^{I}_{3}\phi^{e,o}_{2}(q)$ from which by $q \to q^{-1}$ the sets ${}^{II}_{3}\phi^{e,o}_{2}(q^{s})$ are obtained, as per the scheme given in figure 2.

Parameters in (28)	Parameters in (49)	Identification (49)
$-n=-R_{2r}$	$A'=-R_{2r}$	$(rst)=(123)$ corresponds to the
$b=-R_{3s}$	$B'=1+R_{2t}$	q-analogue of Raynal $F_p(0; 25)$
$c=-R_{1t}$	$C'=1+R_{3r}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{2t}-R_{3s}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{1r}+R_{3r}$	
$-n=-R_{3s}$	$A'=-R_{3s}$	$(rst)=(123)$ corresponds to the
$b=-R_{1t}$	$B'=1+R_{1s}$	q-analogue of Raynal (16)
$c=-R_{2r}$	$C'=1+R_{3r}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{2t}-R_{3s}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{1s}+R_{2s}$	
$-n=-R_{1t}$	$A'=-R_{1t}$	$(rst)=(123)$ corresponds to the
$b=-R_{2r}$	$B'=1+R_{1s}$	q-analogue of Raynal $F_p(0; 35)$
$c=-R_{3s}$	$C'=1+R_{2t}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{2t}-R_{3s}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{2t}+R_{3t}$	
$-n=-R_{2r}$	$A'=-R_{2r}$	$(rst)=(123)$ corresponds to the
$b=-R_{3s}$	$B'=1+R_{1r}$	q-analogue of Raynal $F_p(0; 24)$
$c=-R_{1t}$	$C'=1+R_{2s}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{3t}-R_{2r}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{1r}+R_{3r}$	
$-n=-R_{3s}$	$A'=-R_{3s}$	$(rst)=(123)$ corresponds to the
$b=-R_{1t}$	$B'=1+R_{3t}$	q-analogue of Raynal (15)
$c=-R_{2r}$	$C'=1+R_{2s}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{3t}-R_{2r}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{1s}+R_{2s}$	
$-n=-R_{1t}$	$A'=-R_{1t}$	$(rst)=(123)$ corresponds to the
$b=-R_{2r}$	$B'=1+R_{3t}$	q-analogue of Raynal (17)
$c=-R_{3s}$	$C'=1+R_{1r}$	
$d=1+R_{3t}-R_{2r}$	$D'=1+R_{3t}-R_{2r}$	
$e=1+R_{2t}-R_{3s}$	$E'=2+R_{2t}+R_{3t}$	

(59a) in reference I can be identified with one member of the first entry in table 3 for $(rst)=(312)$ in the parameter set given in column 2 for the ${}^{I}_{3}\phi^{e}_{2}(q)$. The other entries in this table have no equivalents in reference I.

We find that the use of *reversal* formula on the sets of three ${}^{I}_{3}\phi^{e,o}_{2}(q)$ and ${}^{II}_{3}\phi^{e,o}_{2}(q^{s})$ of the form (47)—obtained by the Weber–Erdelyi transformation II (27) on the Van der Waerden sets given by (37) and (37′)—leads one to sets of three ${}^{II}_{3}\phi^{o,e}_{2}(q^{s})$ and ${}^{I}_{3}\phi^{o,e}_{2}(q)$ of the form (49), obtained by KTW transformation (28) on the Van der Waerden sets (37) and (37′). This is schematically shown in figure 3.

Thus, to summarize (as far as the identification of the results in our paper with the the ones in reference I goes), in tables 1-3 are given 12 sets each of ${}^{I}_{3}\phi^{e,o}_{2}(q)$ and

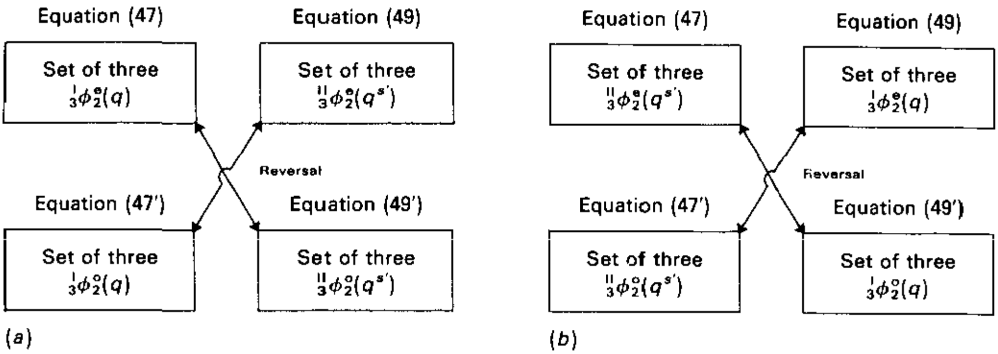

Figure 3. Role of *reversal* on (47), (47′) resulting in (49′), (49).

${}^{\mathrm{II}}_{3}\phi^{e,o}_{2}(q^{s})$, where e and o represent even and odd permutations of (123). All these 12 sets were generated from the given Van der Waerden sets of ${}^{\mathrm{I}}_{3}\phi^{e,o}_{2}(q)$ and ${}^{\mathrm{II}}_{3}\phi^{e,o}_{2}(q^{s})$. In all, we have therefore listed 156 ${}_{3}\phi_{2}$ forms for the $q-3$-j coefficient, of which seven ${}_{3}\phi_{2}$ forms alone are given in reference I.

The middle columns of tables 1-3, as well as the expressions (45) and (45′), (47) and (47′), (49) and (49′) reveal their invariance under $A'\leftrightarrow C'$; $A'\leftrightarrow B'$ and $D'\leftrightarrow E'$; $B'\leftrightarrow C'$, respectively. As in Rajeswari and Srinivasa Rao (1989), only the Van der Waerden forms (37), (39), (43) and (44) clearly exhibit the $S_3\times S_2$ symmetry due to their invariance under the 3! numerator and 2! denominator parameter permutations.

5. *q*-analogues of the 6-*j* coefficients

Kirillov and Reshetikhin (1988) have explicitly derived an expression for the $q-6$-j coefficient by generalizing the procedure of Racah (1942) and obtained

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{\mathrm{R}} = (-1)^{a+b+c+d}\Delta_{\mathrm{R}}(abe)\Delta_{\mathrm{R}}(cde)\Delta_{\mathrm{R}}(acf)\Delta_{\mathrm{R}}(bdf)$$

$$\times\sum_p(-1)^p[p+1]!\left(\prod_{i=1}^{4}[p-\alpha_i]!\prod_{j=1}^{3}[\beta_j-p]!\right)^{-1} \tag{51}$$

where

$$\alpha_1=a+b+e \qquad \alpha_2=c+d+e \qquad \alpha_3=a+c+f \qquad \alpha_4=b+d+f$$

$$\beta_1=a+b+c+d \qquad \beta_2=a+d+e+f \qquad \beta_3=b+c+e+f$$

and

$$\max(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leqslant p\leqslant\min(\beta_1,\beta_2,\beta_3) \tag{52}$$

and $\Delta_{\mathrm{R}}(xyz)$ is defined in (31). In their symmetric notation (4), for the basic number, by simply replacing the q-factorials by ordinary factorials in the derived expressions for the $q-6$-j coefficient, the known expression for the 6-j coefficient can be obtained. This felicity does not pertain to the $q-3$-j coefficient, even for special values of its arguments. It is necessary to resort to the asymmetric Heine notation (3), to enable one to write the $q-6$-j coefficient as a basic hypergeometric series. In the Heine notation (51) becomes

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = N \sum_p (-1)^p [p+1]! \left(\prod_{i=1}^{4} [p-\alpha_i]! \prod_{j=1}^{3} [\beta_j - p]! \right)^{-1} \tag{53}$$

where

$$N = q^{-(1/4)[2\beta_1(\beta_1-1)+2\beta_2(\beta_2-1)+2\beta_3(\beta_3-1)]} q^{(1/4)(\beta_1+\beta_2+\beta_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$
$$\times q^{-(1/4)[\alpha_1(\alpha_1+1)+\alpha_2(\alpha_2+1)+\alpha_3(\alpha_3+1)+\alpha_4(\alpha_4+)]} \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf). \tag{54}$$

Substituting $n = \beta_j - p$ $(j = 1, 2, 3)$ in (53) and using the same procedure adopted in the case of the $q-3$-j coefficient, we get

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = (-1)^{E+1} M q^P \Gamma_q[1-E; 1-A, 1-B, 1-C, 1-D, F, G] \times {}_4\phi_3 \begin{bmatrix} A, B, C, D \\ E, F, G \end{bmatrix}; q, q \Bigg] \tag{55}$$

where

$$A = -R_{1p} \qquad B = -R_{2p} \qquad C = -R_{3p} \qquad D = -R_{4p}$$
$$E = -R_{1p} - R_{2p} - R_{3q} - R_{4r} - 1 \qquad F = 1 + R_{3q} - R_{3p}$$
$$G = 1 + R_{4r} - R_{4p} \qquad M = \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf),$$

and

$$P = \tfrac{1}{4}[(E+1)(E+2) - (F-E-2)(F-E-3) - (G-E-2)(G-E-3)] + \tfrac{1}{4}(F+G-E-3)^2. \tag{56}$$

In (56), the R_{ik} represent the elements of the Bargmann (1962) and Shelepin (1964) 4×3 symbol:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \begin{Vmatrix} \beta_1-\alpha_1 & \beta_2-\alpha_1 & \beta_3-\alpha_1 \\ \beta_1-\alpha_2 & \beta_2-\alpha_2 & \beta_3-\alpha_2 \\ \beta_1-\alpha_3 & \beta_2-\alpha_3 & \beta_3-\alpha_3 \\ \beta_1-\alpha_4 & \beta_2-\alpha_4 & \beta_3-\alpha_4 \end{Vmatrix} = \|R_{ik}\|. \tag{57}$$

It is to be noted that for cyclic permutations of $(pqr) = (123)$, we obtain the set I of three ${}_4\phi_3(q)$ functions which in the limit $q \to 1$ results in the set I of three ${}_4F_3(1)$ functions for the 6-j coefficient (Srinivasa Rao *et al* 1975).

Substituting $n = p - \alpha_i$, $(i = 1, 2, 3, 4)$ in (53) and adopting the same procedure, we get

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = (-1)^{A'} M q^{P'} \Gamma_q[A'; 1-B', 1-C', 1-D', E', F', G'] \times {}_4\phi_3 \left[\begin{matrix} A', B', C', D' \\ E', F', G' \end{matrix} ; q, q \right] \tag{58}$$

where

$$A' = R_{q2} + R_{r1} + R_{s3} + 2 \qquad B' = -R_{p1} \qquad C' = -R_{p2} \qquad D' = -R_{p3}$$
$$E' = R_{q1} - R_{p1} + 1 \qquad F' = R_{r1} - R_{p1} + 1 \qquad G' = R_{s1} - R_{p1} + 1$$
$$P' = -\tfrac{1}{4}[(A'-B'-2)(A'-B'-3) + (A'-C'-2)(A'-C'-3) + (A'-D'-2)(A'-D'-3)]$$
$$+\tfrac{1}{4}[(A'-B'-C'-D'-2)^2 + 2(A'-2)(A'-3)] \qquad (59)$$

for cyclic permutations of $(pqrs) = (1234)$. In the limit $q \to 1$, this set of four ${}_4\phi_3(q)$ functions reduce to the set II of four ${}_4F_3(1)$ functions for the 6-j coefficient (Srinivasa Rao and Venkatesh 1977).

The expressions (55), (56) are invariant under the permutation of A, B, C, D and F, G, so that each one of the three basic hypergeometric series belonging to set I accounts for 48 symmetries of the $q-6$-j coefficient. Similarly, the expressions (58), (59) exhibit invariance under the permutation of B', C', D' and E', F', G', so that each one of the four basic hypergeometric series belonging to set II accounts for 36 symmetries of the $q-6$-j coefficient. Thus, these equivalent sets are necessary and sufficient to account for the 144 symmetries of the $q-6$-j coefficient.

Unlike the case of the $q-3$-j coefficient, the basic hypergeometric functions occurring in (55) and (59) are Saalschutzians since the numerator and denominator parameters satisfy the conditions

$$A + B + C + D + 1 = E + F + G$$
$$A' + B' + C' + D' + 1 = E' + F' + G'. \qquad (60)$$

Due to this property, when the *reversal* formula (18) is used in (55) or (58), the basic hypergeometric series which is a polynomial in q transforms into a polynomial in q^s or $q^{s'}$ but due to the Saalschutzian condition (60), $s = E + F + G - A - B - C - D = 1$ and $s' = E' + F' + G' - A' - B' - C' - D' = 1$. The fact $q^s = q$ also is a pointer to a simplification in the structure of the q-generalization of the 6-j coefficient in terms of basic hypergeometric series. It is straightforward to show that (55) and 58), like the Kirillov-Reshetikhin formula for the $q-6$-j coefficient, are invariant under $q \to q^{-1}$ transformation so that

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{q^{-1}}. \qquad (61)$$

After simplifications it can be shown that *reversal* (18) transforms set I (55) into set II (58) and vice versa.

In (55), when $(pqr) = (123)$, use of the ${}_4\phi_3(q)$ transformation (26) results in the *new* expression for the Racah coefficient given by (11) of Kachurik and Klimyk (1990). In fact, this latter equation itself can be shown to be a q-generalization of the formula (17) of Raynal (1979)—on which are superimposed column and 'row' permutations to get

$$\begin{Bmatrix} c & e & d \\ f & b & a \end{Bmatrix}.$$

It is to be noted that while a ${}_4\phi_3(q)$ belonging to set I or set II accounts for 48 or 36 respectively of the 144 symmetries of a $q-6$-j coefficient, the *new* ${}_4\phi_3(q)$ form given by Kachurik and Klimyk (1990) exhibits only eight symmetries, since permutation of a positive parameter with a negative parameter will not yield a known, meaningful symmetry (see Srinivasa Rao *et al* 1975).

6. Summary

q-generalizations of the set of six ${}_3F_2(1)$ functions, necessary and sufficient to account for the 72 symmetries of the $q-3$-j coefficient have been derived. The structure of the q-generalization of the set has been analysed in terms of the reversal and $q \to q^{-1}$ transformations. From the symmetric Van der Waerden form for the set of six ${}_3\phi_2$ functions, using the q-analogues of the Erdelyi–Weber *transformation* I, the q-analogues of the Racah, Wigner and Majumdar sets were obtained. Use of the q-analogue of Erdelyi–Weber transformation II and the Kummer–Thomae–Whipple transformation on the Van der Waerden set of ${}_3\phi_2$ resulted in nine sets of ${}_3\phi_2$ forms, one member of each set being a q-analogue of a ${}_3F_2(1)$ form found by Raynal (1978). Of the 156 ${}_3\phi_2$ forms listed in tables 1-3 for the $q-3$-j coefficient, seven ${}_3\phi_2$ forms are also given by Groza *et al* (1990).

The $q-6$-j coefficient, unlike the $q-3$-j coefficient, exhibits $q \to q^{-1}$ symmetry. The q-generalizations of the sets I and II of three and four ${}_4F_3(1)$ functions, respectively, necessary and sufficient to account for the 144 symmetries of the $q-6$-j coefficient have been derived. These two sets are related to each other by the *reversal* of series. Given any one of the ${}_4\phi_3$ belonging to either of the sets, all the elements of the other set can be obtained using the *reversal* formula. A member belonging to set I of ${}_4\phi_3(q)$s for the $q-6$-j coefficient, when transformed by (26), yields the ${}_4\phi_3$ form given by Kachurik and Klimyk (1990), which is a q-generalization of formula (17) in Raynal (1979).

The sets of ${}_3\phi_2$ forms and the sets of ${}_4\phi_3$ forms for the $q-3$-j and the $q-6$-j coefficients given here thus reveal the full structure of the q-generalizations of the 3-j and the 6-j coefficients and contribute to the complete understanding of this aspect of the Racah–Wigner algebra of $SU_q(2)$.

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